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THE JOHNS HOPKINS UNIVERSITY
APPLIED PHYSICS LABORATORY
8621 Georgia Avenue, Silver Spring, Maryland

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THE JOHNS HOPKINS UNIVERSITY
APPLIED PHYSICS LABORATORY
8621 GEORGIA AVENUE SILVER SPRING, MARYLAND

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

ABSTRACT

The equation for axi-symmetric magnetohydrostatic equilibria in stationary conductive fluid is re-examined. Spheroidal magnetic conducting vortices are shown as examples of equilibrium configurations. Then a circulating sphere of viscous conducting fluid moving slowly along an aligned spatially uniform magnetic field, varying linearly with time, is shown to be a megnetohydrodynamic vortex which satisfies all reasonable boundary conditions for real fluids. The possibility of experimental verification is discussed.

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LIST OF SYMBOLS

$\vec{B}$	Magnetic vector in material
H_0	Characteristic intensity of applied magnetic field
$\vec{h}$	Normalized magnetic vector in free space
$\mathscr{h}_i$	Euclidean metric coefficients, $i = 1, 2, 3$
$\hat{1}$	Unit vector
$\vec{J}$	Volume current vector
L	Characteristic length
M	Hartmann number $= \mu H_0 L \sqrt{\sigma/\rho\nu}$
MHD	Shorthand for magnetohydrodynamic(s)
p	Scalar pressure
$\vec{q}$	Normalized velocity vector
r	Radial distance from origin
Re	Reynolds number $= U L \nu^{-1}$
R_m	Magnetic Reynolds number $= \mu \sigma L U$
t	Time
u, v	Normalized velocity components
U	Characteristic velocity magnitude
x	Axial distance from origin
$C_n^{-1/2}(z)$	Gegenbauer polynomial
$D_n^{-1/2}(z)$	Gegenbauer function of the second kind
$\mathbb{D}$	"Stokesian" operator, see Eq. (7c).
$\mathcal{L}_{ij}$	Linear partial differential operator
$Q_1(z)$	Bessel function of second kind
$Q_1^1(z)$	Associated Bessel function
∇^2_{axi}	Axi-symmetric Laplacian operator

LIST OF SYMBOLS (Cont'd)

α, β, φ	Axi-symmetric coordinate system
β	Angle of normal to α -isovalue with respect to axis
Δ	Determinant for solution of simultaneous equations
η	Oblate spheroidal coordinate
θ	Polar angle (spherical polar coordinates)
μ	Magnetic permeability
ν	Kinematic viscosity of fluid
ξ	Prolate spheroidal coordinate
ρ	Density of fluid
σ	Electrical conductivity of fluid
φ	Azimuthal angle
ψ	Flow streamfunction
Ψ	Magnetic flux function
Ψ_{ij}	Magnetohydrostatic flux function
ϖ	Distance from axis
$\vec{\omega}$	Vorticity vector, $\nabla \times \vec{q}$

Subscripts

$(\ )_\alpha$	Component of vector in α -direction
$(\ )_{in}$	Internal
$(\ )_{ex}$	External

ON MAGNETOHYDRODYNAMIC VORTICES

I. INTRODUCTION

Magnetohydrodynamics (MHD) deals with the dynamic behavior of electrically conducting fluids under the influence of electric and/or magnetic fields. Just as in fluid dynamics alone, few of the infinite number of possible MHD configurations lend themselves to ready analysis. This report, continuing a line of investigation of viscous incompressible conducting fluids (Refs. 1, 2 and 3)*, deals with magnetohydrodynamic vortices.

It will be assumed that the fluid motions postulated are steady and that the configurations are axi-symmetric. The main advantage of the axi-symmetry is to make the governing equations more tractable without sacrificing physical reality. Axi-symmetrical configurations also permit easier design of laboratory apparatus to test the theory. It remains to be seen if the corresponding flow patterns of real conductive fluids are steady, because experimental investigations of plasmas** have shown a wide variety of instability modes (Ref. 4).

Some previous workers have investigated magnetohydrostatic equilibria, that is, possible closed magnetic configurations of stationary conducting fluids (Ref. 5). Their results are extended here with an infinite class of spheroidal magnetic vortices. A truly magnetohydrodynamic vortex, a spherical conducting, circulating vortex in dynamic equilibrium is also demonstrated. Qualitative similarities to MHD vortices actually observed, such as "ball lightning" and "plasma vortex rings", will be discussed.

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* References are on pages 38 to 40.

** Plasmas are gases containing ionized (or charged) particles as well as neutral particles as in ordinary fluids, but in some features of their behavior they can be considered as incompressible homogeneous fluids.

II. ANALOGY OF MAGNETIC VORTICES TO FLUID VORTICES

The governing equations for the MHD problem are the combination of Maxwell's electromagnetic homogeneous field equations and the fluid dynamic equations with the interaction terms included. These equations are found in a number of current textbooks on the subject (Refs. 5, 6, 7), are given in the previous reports (Refs. 1, 2) and will not be repeated here. It is assumed that there are no free charges* so there are no electrostatic forces. The notation is fairly standard (see the table of symbols, page iv).

For a magnetic vortex of stationary conductive fluid the force equilibrium requires (Ref. 8)

$$\nabla p = \vec{J} \times \vec{B} \tag{1a}$$

where p is the scalar pressure, including the "magnetic pressure" $B^2/4\pi$, $\vec{J}$ (e.m.u.) are the volume currents and $\vec{B}$ (e.m.u.) the magnetic field vectors. For hydrostatic equilibrium the surfaces of constant pressure hold both the magnetic field lines and the current vectors. Such surfaces are sometimes called "magnetic surfaces" (Ref. 5). Since

$$\vec{J} \propto \nabla \times \vec{B} \tag{2}$$

the equation above for magnetohydrostatic equilibrium is formally the same as the momentum equation for steady non-magnetic flow of a perfect fluid,

$$\nabla \times (\vec{q} \times \vec{q}) = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\vec{q}\cdot\vec{q}\right) = \nabla\tilde{p} \tag{3a}$$

with the substitution of $\vec{B}$ for the velocity vector $\vec{q}$, $\nabla \times \vec{q} = \vec{\omega}$ the vorticity vector for $\nabla \times \vec{B} = \vec{J}$ and p for $\tilde{p}$. Taking the curl

* For plasmas this amounts to no charge separation effects between the ion species. A mass of the plasma is electrically neutral.

of Eqs. (1a) and (3a) reduces them to

$$\nabla \times \vec{J} \times \vec{B} = 0. \tag{1b}$$

$$\nabla \times \vec{\omega} \times \vec{q} = 0. \tag{3b}$$

If the vector fields $\vec{B}$ or $\vec{q}$ are simply axi-symmetric without azimuthal components, the vector equations reduce to scalar equations.

A number of equivalent scalar functions can be defined for axi-symmetric configurations. As in a previous report (Ref. 2), it is convenient to use a flow streamfunction ψ and a magnetic flux function Ψ. The streamfunction ψ is introduced in terms of the normalized velocity components ($u_r, v_\theta, 0$) for a spherical polar coordinate system (r , θ , φ):

$$u_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \quad ; \quad v_\theta = \frac{1}{r \sin\theta}\frac{\partial\psi}{\partial r} \quad . \tag{4}$$

Likewise, a magnetic flux function Ψ is introduced in terms of the normalized magnetic vector components:

$$h_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\Psi}{\partial\theta} \quad ; \quad h_\theta = \frac{1}{r \sin\theta}\frac{\partial\Psi}{\partial r} \quad . \tag{5}$$

The normalizing factor is H_o, a characteristic magnetic field intensity, and

$$\vec{B} = \mu H_o \vec{h} = \mu H_o (h_r \hat{i}_r + h_\theta \hat{i}_\theta). \tag{6}$$

The curl of the steady MHD momentum equation is (Ref. 2, Eq. (20))

$$Re \left\{ \frac{-1}{r^2 \sin\theta} \frac{\partial(\psi, D\psi)}{\partial(r,\theta)} + \frac{2D\psi}{r^2 \sin^2\theta}\left(\cos\theta \frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right)\right\}$$
$$= D^2\psi - \frac{M^2}{R_m}\sin\theta \frac{\partial(\Psi, D\Psi/r^2\sin^2\theta)}{\partial(r,\theta)} \quad . \tag{7a}$$

which can be rewritten

$$\sin\theta \, \frac{\partial(\psi, \mathbb{D}\psi/r^2\sin^2\theta)}{\partial(r,\theta)} = \frac{1}{Re}\mathbb{D}^2\psi - \frac{M^2}{Re\, R_m} \frac{\partial(\Psi, \mathbb{D}\Psi/r^2\sin^2\theta)}{\partial(r,\theta)} \sin\theta. \tag{7b}$$

Here, the non-dimensional parameters are

$M = \mu H_0 L \sqrt{\frac{\sigma}{\rho\nu}}$, Hartmann number

$Re = U L \nu^{-1}$, Reynolds number

$R_m = \mu U \sigma L$, Magnetic Reynolds number,

and

$$\mathbb{D} = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{7c}$$

For no magnetic field, $M = 0$, and perfect (inviscid) fluid implies $Re = \infty$. The vorticity equation, Eq. (7b), becomes

$$\frac{\partial(\psi, \mathbb{D}\psi/r^2\sin^2\theta)}{\partial(r,\theta)} = 0. \tag{8}$$

For stationary fluid, $\psi = 0$, and Eq. (7b) reduces to

$$\frac{\partial(\Psi, \mathbb{D}\Psi/r^2\sin^2\theta)}{\partial(r,\theta)} = 0. \tag{9}$$

These are the scalar equivalents to Eqs. (3b) and (1b) respectively. The correspondence of perfect fluid streamfunction ψ to the magnetohydrostatic flux function Ψ is pointed out as part of the analogy between steady perfect fluid flow and magnetohydrostatics. Just as $\psi =$ constant can be taken as a streamsurface or flow current flux boundary, $\Psi =$ constant is a "magnetic surface" or a magnetic flux boundary (see Fig. 1).

In general, the Jacobian, Eq. (8) or (9), leads to a non-linear equation in ψ (or Ψ), but for the particular case

$$\frac{\mathbb{D}\psi}{r^2\sin^2\theta} = C_0, \text{ a constant}, \tag{10}$$

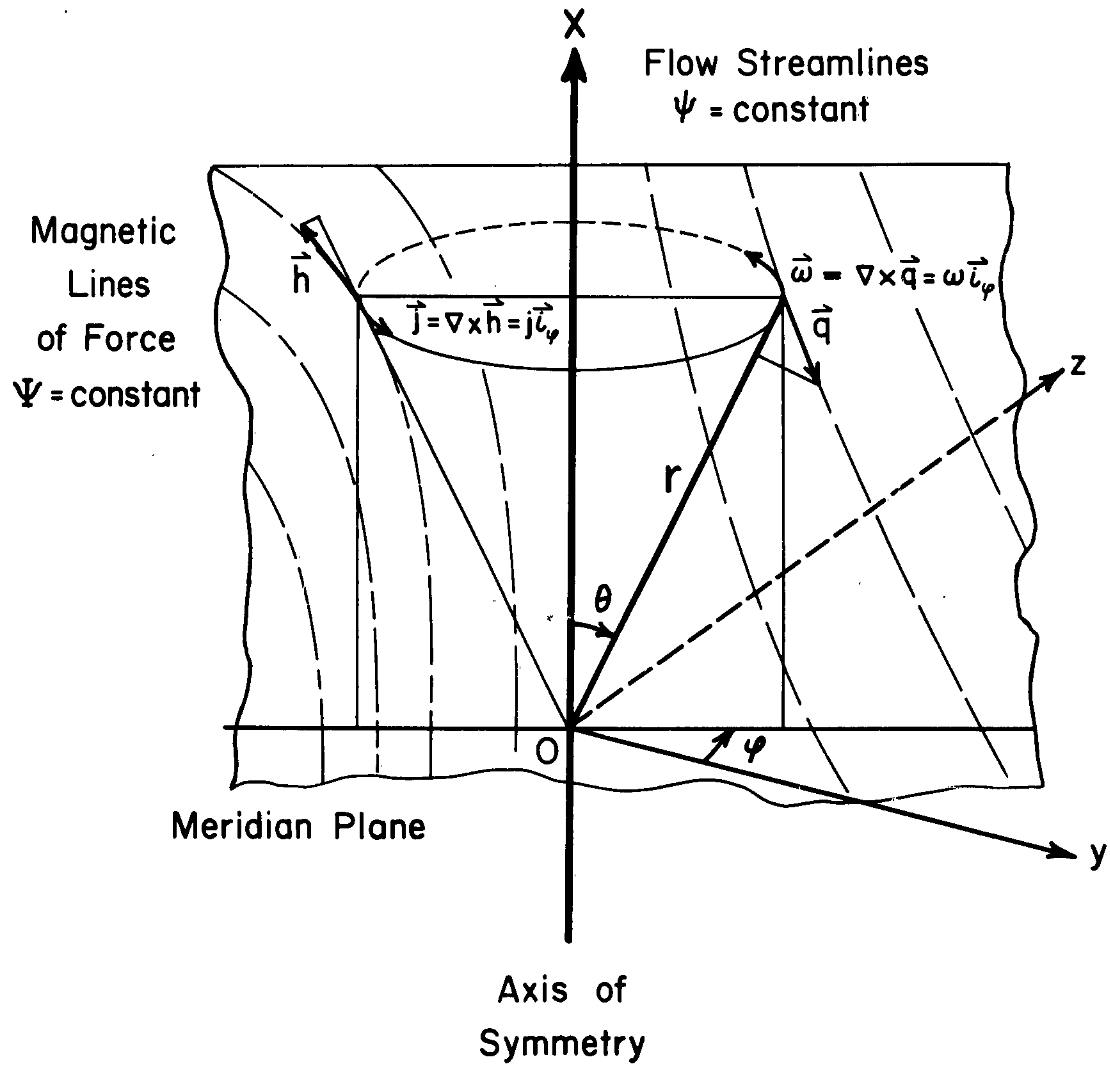

Fig. 1 COORDINATES AND AXI-SYMMETRIC VELOCITY AND MAGNETIC VECTOR FIELDS

the Jacobian is automatically zero. Then the vorticity magnitude $|\vec{\omega}|$, $\frac{1}{r}\frac{\partial}{\partial r}(r V_\theta) - \frac{1}{r}\frac{\partial}{\partial \theta}(u_r) = \frac{D\Psi}{r \sin\theta}$, is proportional to $(r \sin\theta)$ the distance from the axis. Such a vorticity distribution has been considered before in both oblate and prolate spheroidal coordinates (Ref. 9). Therefore, by analogy, magnetic spheroidal vortices of conducting fluid with magnetic surfaces $\Psi = 0$ at $\eta = \eta_0$ or $\xi = \xi_0$ are given by

Oblate: $$\Psi_{in} = C_1 \sin^2\beta \cosh^2\eta \left\{\sinh^2\eta_0 - \sinh^2\eta + \frac{\cos^2\beta}{\sinh^2\eta_0}(\sinh^2\eta_0 - \sinh^2\eta)\right\}. \tag{11a}$$

Prolate: $$\Psi_{in} = C_2 \sin^2\beta \sinh^2\xi \left\{\sinh^2\xi_0 - \sin^2\beta \sinh^2\xi - \cos^2\beta \cosh^2\xi_0 \tanh^2\xi_0\right\}. \tag{11b}$$

In the limit of fineness ratio one, or a spherical vortex of unit radius

$$\Psi_{in} = C_3 \sin^2\theta \, (r^2 - r^4) \tag{12}$$

which is the Hill vortex solution for a perfect fluid (Ref. 10).

Assume that outside the vortex boundary the fluid is non-conductive. Then $\vec{J} = 0$ or $D\Psi = 0$. By analogy with irrotational flow (Ref. 11), solutions with uniform magnetic field at infinity and $\Psi = 0$ on spheroidal boundaries can be written

Oblate: $$\Psi_{ex} = \frac{-\sin^2\beta}{2}\left[\cosh^2\eta - \frac{D_2^{-1/2}(i \sinh\eta)}{D_2^{-1/2}(i \sinh\eta_0)} \cosh^2\eta_0\right]. \tag{13a}$$

Prolate: $$\Psi_{ex} = \frac{-\sin^2\beta}{2}\left[\sinh^2\xi - \frac{D_2^{-1/2}(\cosh\xi)}{D_2^{-1/2}(\cosh\xi_0)} \sinh^2\xi_0\right]. \tag{13b}$$

Here $D_n^{-1/2}(z)$ is a Gegenbauer function of the second kind of order n and index ν. In the limit of a spherical boundary $r = 1$

$$\Psi_{ex} = \frac{-\sin^2\theta}{2}\left[r^2 - \frac{1}{r}\right]. \tag{14}$$

Besides the requirement that the boundary surface be a magnetic surface, there is the further requirement with finite electrical conductivities that the tangential components of the magnetic field be continuous (Ref. 12). (Physically impossible infinite electrical conductivity is often assumed in MHD.)

That is, $$\vec{n} \times \left[(\mu_{in} \vec{h}_{in}) - \mu_{ex} \vec{h}_{ex} \right] = 0,$$

which reduces to

$$\mu_{ex} \frac{\partial \Psi_{ex}}{\partial n} = \mu_{in} \frac{\partial \Psi_{in}}{\partial n} , \tag{15}$$

with $\vec{n}$ the normal to the boundary. Only in the case of the sphere can this requirement be met directly with the given magnetic flux functions. Thus, $C_3 = \frac{3}{4} \frac{\mu_{ex}}{\mu_{in}}$ and the normalized magnetic components for the spherical vortex, Fig. 2a, are

Inside (Ψ_{in})	Outside (Ψ_{ex})
$h_r = -\frac{3}{2} \frac{\mu_{ex}}{\mu_{in}} \cos\theta\,(1-r^2)$	$h_r = \cos\theta\,(1 - \frac{1}{r^3})$
$h_\theta = \frac{3}{2} \frac{\mu_{ex}}{\mu_{in}} \sin\theta\,(1-2r^2)$	$h_\theta = -\sin\theta\,(1 + \frac{1}{2r^3})$ (16)
$j_\varphi = 15 \frac{\mu_{ex}}{\mu_{in}} r \sin\theta$	$j_\varphi = 0.$

This result checks that of Ref. 13 where $\mu_{ex} = \mu_{in}$. The internal magnetic field of the conducting sphere contrasts with the uniform internal magnetic field induced in a non-conducting sphere located in an infinite uniform magnetic fields; see Ref. 11 and Fig. 2b.

If we admit a discontinuity in the tangential components, as is often done in magnetic field problems (Ref. 12), the spheroidal magnetic vortices are acceptable magnetohydrostatic equilibrium configurations, but with surface currents. These vortices require additional surface current distributions on the spheroidal boundary* as well as the volume currents within:

* The Russian author Shafranov (Ref. 13), after demonstrating the spherical magnetic vortex, claimed one could easily obtain the analogous spheroidal configuration. Since he allowed surface currents in other configurations in the paper, the spheroidal vortices here are probably what he had in mind. So far as it is known, they have never been published.

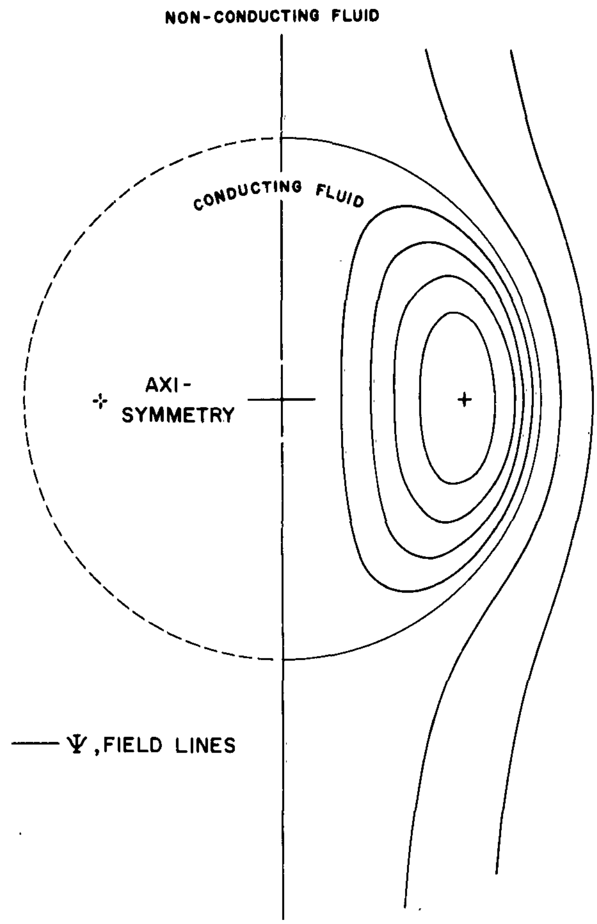

Fig. 2a SPHERICAL MAGNETIC VORTEX IN UNIFORM MAGNETIC FIELD

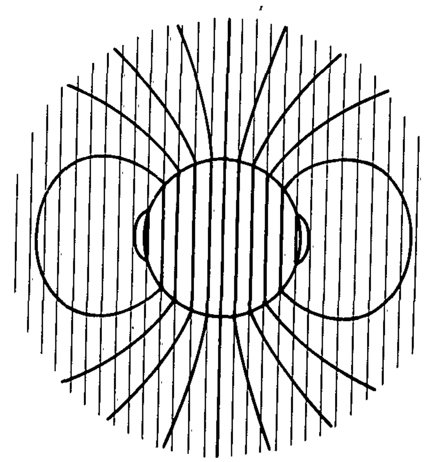

Fig. 2b MAGNETIC FIELD LINES FOR A SPHERE IN A UNIFORM MAGNETIC FIELD (MAGNETOSTATIC WITH DIFFERING PERMEABILITIES)

Oblate: $j_\varphi = 10 C_1 \sin\beta \cosh\eta,$

where C_1 is the coefficient in Eq. (11a) and

$$C_1 = \frac{1}{2}\frac{\mu_{ex}}{\mu_{in}}\left\{\frac{1}{\cosh^2\eta_0}\left(1 - \frac{2\coth\eta_0\, Q_1(i\sinh\eta_0)}{Q_1'(i\sinh\eta_0)}\right)\right\}. \tag{17a}$$

Prolate: $j_\varphi = 10 C_2 \sin\beta \sinh\xi,$

where C_2 is given by

$$C_2 = \frac{1}{2}\frac{\mu_{ex}}{\mu_{in}}\left\{\frac{1}{\sinh^2\xi_0}\left(1 - \frac{2\tanh\xi_0\, Q_1(\cosh\xi_0)}{Q_1'(\cosh\xi_0)}\right)\right\}. \tag{17b}$$

The jump in tangential components at the spheroid surface, usually represented as a surface density K due to surface currents, is illustrated in Fig. 3 for several values of η_0.

If we do not admit the presence of surface currents, the exact ellipsoidal boundary shape is not an equilibrium configuration. However, it seems possible to modify the boundary to meet the tangential condition. It will be assumed that the boundary is only a small deviation from an exact spheroid so we can set the boundary conditions on the spheroid surface rather than the deformed surface.

The gradients of Ψ do not match on the spheroid because of the angle dependent term within the bracket of Eq. (11a) and Eq. (11b). We can add arbitrary amounts of irrotational Ψ with the same angular dependence to both Ψ_{ex} and Ψ_{in}. These deformation flux functions will be designated by primes so

Oblate:

$$\begin{aligned} \Psi'_{in} &= A\, C_2^{-1/2}(\cos\beta)\, C_2^{-1/2}(i\sinh\eta) + B\, C_4^{-1/2}(\cos\beta)\, C_4^{-1/2}(i\sinh\eta), \\ \Psi'_{ex} &= a\, C_2^{-1/2}(\cos\beta)\, D_2^{-1/2}(i\sinh\eta) + b\, C_4^{-1/2}(\cos\beta)\, D_4^{-1/2}(i\sinh\eta). \end{aligned} \tag{18a}$$

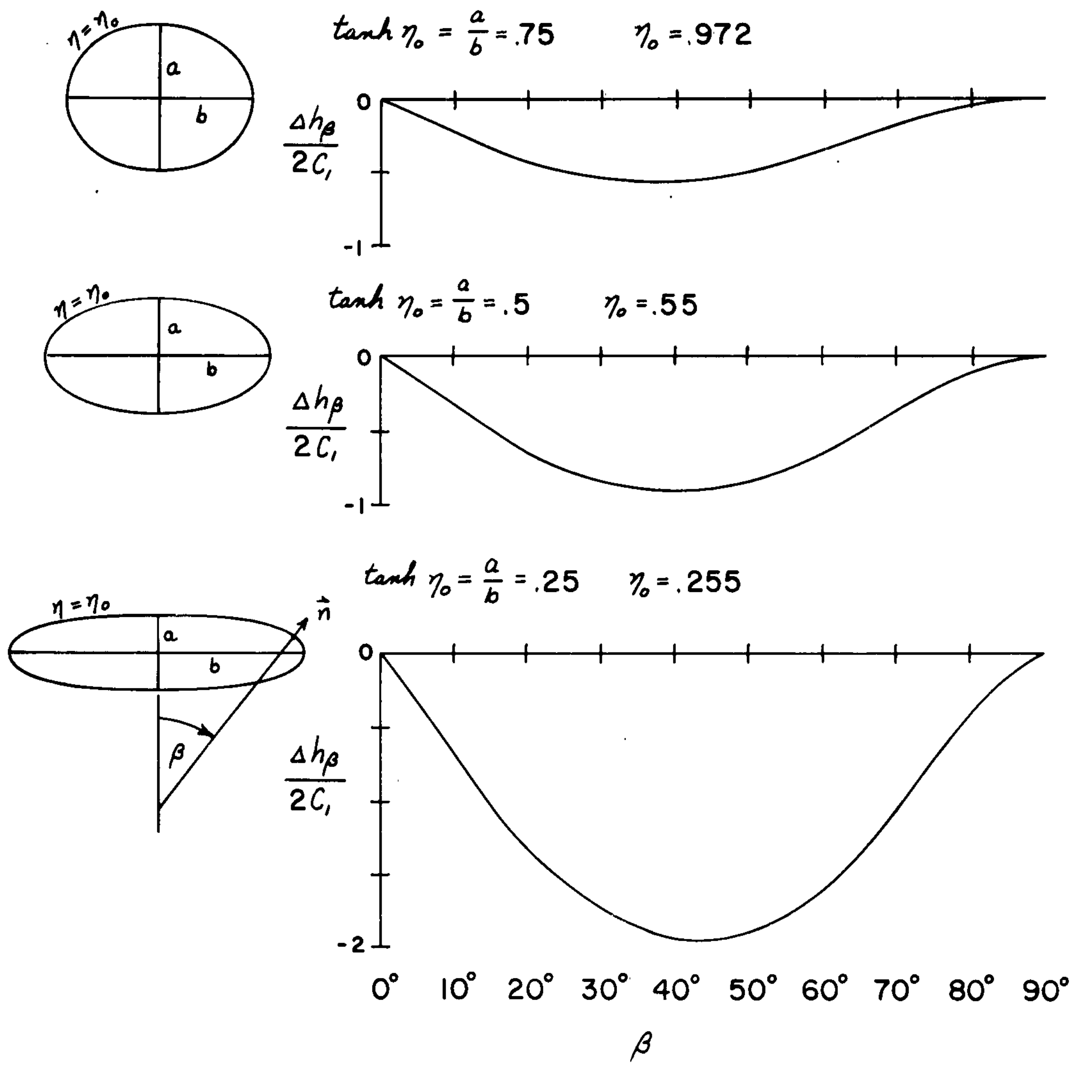

Fig. 3 JUMP IN TANGENTIAL MAGNETIC VECTOR COMPONENT FOR SEVERAL SPHEROIDS

Prolate:

$$\Psi'_{in} = A' C_2^{-1/2}(\cos\beta)\, C_2^{-1/2}(\cosh\xi) + B' C_2^{-1/2}(\cos\beta)\, C_4^{-1/2}(\cosh\xi),$$

$$\Psi'_{ex} = a' C_2^{-1/2}(\cos\beta)\, D_2^{-1/2}(\cosh\xi) + b' C_4^{-1/2}(\cos\beta)\, D_4^{-1/2}(\cosh\xi). \qquad (18b)$$

These solutions have been selected from Table I in Ref. 3 (or Table I in Ref. 11 with k = - 1). The respective internal and external functions remain finite at the origin and go to zero at infinity.

The steps in the evaluation of the constants for the deformed boundary will only be carried through in detail for the oblate spheroid, but it is obvious that a similar process can be carried out for the prolate one as well. Writing out the Gegenbauer polynomials for ($\cos\beta$) and grouping

Oblate:

$$\Psi'_{in} = \frac{\sin^2\beta}{2}\left\{\cosh^2\eta\left[\frac{A}{2} + \frac{B}{64}(5\sinh^2\eta+1) - \frac{5B\cos^2\beta}{64}(5\sinh^2\eta+1)\right]\right\},$$

$$\Psi'_{ex} = \frac{\sin^2\beta}{2}\left\{a\, D_2^{-1/2}(i\sinh\eta) + \frac{b}{8} D_4^{-1/2}(i\sinh\eta) - \frac{5b}{8}\cos^2\beta\, D_4^{-1/2}(i\sinh\eta)\right\}. \qquad (19)$$

Set A and a so that the function in the bracket reduces to a function of ($\cos^2\beta$) at $\eta = \eta_0$.

$$A = -\frac{5\sinh^2\eta_0+1}{32} B \quad ; \quad a = -\frac{D_4^{-1/2}(i\sinh\eta_0)}{8 D_2^{-1/2}(i\sinh\eta_0)} b . \qquad (20)$$

Now at $\eta = \eta_0$ require continuity of total Ψ and Eq. (15).

$$\Psi_{in} + \Psi'_{in} = \Psi_{ex} + \Psi'_{ex},$$

$$\mu_{in}\frac{\partial}{\partial\eta}(\Psi_{in} + \Psi'_{in}) = \mu_{ex}\frac{\partial}{\partial\eta}(\Psi_{ex} + \Psi'_{ex}). \qquad (21)$$

Substituting into these expressions, the following three equations for the three constants B, b and C_1 are obtained.

$$\frac{B}{8}\cosh^2\eta_0(5\sinh^2\eta_0+1) = b\, D_4^{-1/2}(i\sinh\eta_0). \qquad (22)$$

$$\mu_{in}\left\{2\cosh^2\eta_0 \sinh\eta \left(C_1 - \frac{25B}{128}\right)\right\}$$
$$= \mu_{ex}\left\{\sinh\eta_0 - \frac{2\cosh\eta_0\, Q_1(i\sinh\eta_0)}{Q_1'(i\sinh\eta_0)} + \frac{b}{8}\,\frac{\left[\frac{\partial}{\partial\eta}\left(D_4^{-1/2}[i\sinh\eta]\right)\right]_{\eta=\eta_0}}{\cosh\eta_0}\right\}. \tag{23}$$

$$\mu_{in}\left\{2C_1\frac{\cosh^3\eta_0}{\sinh\eta_0} + \frac{5B}{128}\left[2\cosh\eta_0\sinh\eta_0(5\sinh^2\eta_0+1) + 10\sinh\eta_0\cosh^3\eta_0\right]\right\}$$
$$= \mu_{ex}\,\frac{5b}{16}\left[\frac{\partial}{\partial\eta}D_4^{-1/2}(i\sinh\eta)\right]_{\eta=\eta_0}. \tag{24}$$

These algebraic equations involve rather complicated expressions in η_0 . Schematically, the equations above can be written

$$f_1(\eta_0)B + f_2(\eta_0)b = 0. \tag{22a}$$

$$g_1(\eta_0)C_1 + g_2(\eta_0)B + g_3(\eta_0)b = g_4(\eta_0). \tag{23a}$$

$$h_1(\eta_0)C_1 + h_2(\eta_0)B + h_3(\eta_0)b = 0. \tag{24a}$$

These equations for the unknowns, C_1, B and b, always possess a solution (unless the determinant of the η_0 function matrix is singular, which can only be true for isolated values of η_0 at most). Denoting the determinant of the coefficients, Δ or

$$\Delta(\eta_0) = \begin{vmatrix} 0 & f_1(\eta_0) & f_2(\eta_0) \\ g_1(\eta_0) & g_2(\eta_0) & g_3(\eta_0) \\ h_1(\eta_0) & h_2(\eta_0) & h_3(\eta_0) \end{vmatrix}. \tag{25}$$

the constants can be written symbolically

$$C_1 = \frac{-g_4(\eta_0)}{\Delta(\eta_0)}\begin{vmatrix} f_1(\eta_0) & f_2(\eta_0) \\ h_2(\eta_0) & h_3(\eta_0) \end{vmatrix}. \tag{26}$$

$$B = \frac{g_4(\eta_0)}{\Delta(\eta_0)} h_1(\eta_0) f_2(\eta_0) . \tag{27}$$

$$b = \frac{-g_4(\eta_0)}{\Delta(\eta_0)} h_1(\eta_0) f_1(\eta_0) = \frac{f_1(\eta_0)}{f_2(\eta_0)} B . \tag{28}$$

So, although the approximately spheroidal solution exists, the deformation of the boundary for a given η_0 requires a bit of numerical calculation. The resulting flux functions will not have $\Psi = 0$ on the spheroid surface $\eta = \eta_0$ but will have $\Psi = 0$ and continuous magnetic components on a deformed surface. These approximately spheroidal surfaces enclose a current bearing region of fluid where the volume currents are proportional to the distance from the axis and there are no surface currents. As η_0 can take on any value, the flux functions represent an infinite class of oblate (almost) spheroidal vortex magnetohydrostatic equilibria. Likewise, there is also the class of prolate approximately spheroidal magnetic vortices in static equilibrium.

Thus Eq. (10) gives rise to an infinite set of axi-symmetric magnetohydrostatic vortices in a spatially uniform magnetic field. However, in order to generate the toroidal currents that characterize these vortices, it is necessary for the intensity of the magnetic field to change with time. From Ohm's law $\vec{J} = \sigma \vec{E}$, the electric field is also toroidal, $\vec{E} = E \hat{\imath}_\varphi$. Without the use of actual voltage sources this electric field can only be generated by a changing magnetic field where $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{dH_0(t)}{dt} \hat{\imath}_x$. J and E vary linearly with distance from the axis which means $\frac{dH_0(t)}{dt}$ is constant or H_0 varies linearly with time.

The magnetohydrostatic vortices are not the only solutions to Eq. (9). In fact, if $D\Psi/r^2\sin^2\theta$ is any function of Ψ, a new set of solutions can be generated. Of course, if the function is Ψ^n with $n > 1$, the equation will be non-linear and difficult to solve. The next section deals with the totality of axi-symmetric magnetohydrostatic equilibria that yield linear equations, including cases with azimuthal magnetic field components.

III. MAGNETOHYDROSTATIC EQUILIBRIA

Essentially the same results may be obtained through the use of any axi-symmetric coordinate system. To take advantage of some previous analysis (Ref. 5), the cylindrical polar coordinate system (x , ϖ , φ) will be used in this section. The flux function Ψ is related to the magnetic components,

$$\varpi h_x = -\frac{\partial \Psi}{\partial \varpi} \quad ; \quad \varpi h_\varpi = \frac{\partial \Psi}{\partial x} \ . \tag{29}$$

Let

$$\varpi h_\varphi = f(\Psi), \tag{30}$$

where f is an arbitrary function of Ψ .

Thompson (Ref. 5) reduces the (x , ϖ) component parts of Eq. (1a) to

$$\frac{\partial p}{\partial \varpi} + \frac{\partial \Psi}{\partial \varpi}\left[\frac{1}{\varpi^2}\mathcal{D}\Psi + \frac{1}{\varpi^2}ff'\right] = 0 , \tag{31}$$

$$\frac{\partial p}{\partial x} + \frac{\partial \Psi}{\partial x}\left[\frac{1}{\varpi^2}\mathcal{D}\Psi + \frac{1}{\varpi^2}ff'\right] = 0 , \tag{32}$$

where $f' = \frac{df}{d\Psi}$ and $\mathcal{D} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial \varpi}$.

Differentiate Eq. (31) with respect to x, Eq. (32) with respect to ϖ and subtract to eliminate p.

$$\frac{\partial \Psi}{\partial \varpi}\frac{\partial}{\partial x}\left[\frac{1}{\varpi^2}\mathcal{D}\Psi + \frac{1}{\varpi^2}ff'\right] - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial \varpi}\left[\frac{1}{\varpi^2}\mathcal{D}\Psi + \frac{1}{\varpi^2}ff'\right] = 0, \tag{33}$$

or

$$\frac{\partial\left(\Psi, \ \frac{\mathcal{D}\Psi + ff'}{\varpi^2}\right)}{\partial(\varpi, x)} = 0 .$$

This Jacobian will be satisfied if

$$\frac{\mathcal{D}\Psi + ff'}{\varpi^2} = -p'(\Psi) = -\frac{dp(\Psi)}{d\Psi}\ , \tag{34}$$

where $p'(\Psi)$ is any function of Ψ and p is pressure (Refs. 5, 13). Thus, if ff'/ϖ^2 and/or $p'(\Psi)$ equals either a constant (including zero) or is proportional to Ψ, Eq. (34) is a linear partial differential equation. All of these classes are summarized in Table I $= \{\mathcal{L}_{ij}\Psi = F_{ij}(\varpi)\}$.

Table I. Table of Linear P. D. E. for Ψ

	p	A	$-B\Psi$	$\frac{1}{2}C\Psi^2$
f	ff' \ $-p'$	0	B	$-C\Psi$
0 a	0	$\mathcal{D}\Psi = 0$	$\mathcal{D}\Psi = B\varpi^2$	$(\mathcal{D}+C\varpi^2)\Psi = 0$
$b\sqrt{\Psi}$	$\frac{b^2}{2}$	$\mathcal{D}\Psi = -\frac{b^2}{2}$	$\mathcal{D}\Psi = B\varpi^2 - \frac{b^2}{2}$	$(\mathcal{D}+C\varpi^2)\Psi = -\frac{b^2}{2}$
$c\Psi$	$c^2\Psi$	$(\mathcal{D}+c^2)\Psi = 0$	$(\mathcal{D}+c^2)\Psi = B\varpi^2$	$(\mathcal{D}+C\varpi^2+c^2)\Psi = 0$

Denote the corresponding solutions $\{\Psi_{ij}\}$. There are four homogeneous equations for Ψ_{11}, Ψ_{13}, Ψ_{31}, and Ψ_{33} with linear operators $\mathcal{L}_{11} = \mathcal{D}$, $\mathcal{L}_{13} = \mathcal{D}+c^2$, $\mathcal{L}_{31} = \mathcal{D}+C\varpi^2$ and $\mathcal{L}_{33} = \mathcal{D}+C\varpi^2+c^2$.
All the remaining equations contain a forcing function as well. In the cylindrical coordinates the operators $\mathcal{D}$ and $\mathcal{D}+c^2$ have been considered long ago for other problems. General separable solutions are known and the functions tabulated. The other homogeneous equations can also be separated and reduced to an unusual second-order ordinary differential equation in ϖ. (See Appendix.) Although the equations here are linear, the full Eq. (9) is not and there can be no superposition of solutions.

Now all these solutions Ψ_{ij} represent a current distribution that satisfies the curl-free condition of $(\vec{J} \times \vec{B})$. Such a distribution is not altered if we describe it in another coordinate system. The set of equations in Table I can be transformed to any axi-symmetric coordinate system (α, β , φ) where the line element is

$$(ds)^2 = h_1^{\,2} d\alpha^2 + h_2^{\,2} d\beta^2 + h_3^{\,2} d\varphi^2 .$$

Then h_3 = distance from the axis = $\varpi(\alpha, \beta)$;

$\mathcal{D} = \frac{h_3}{h_1 h_2}\left\{ \frac{\partial}{\partial\alpha}\left(\frac{h_2}{h_3 h_1}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial\beta}\right)\right\}$, the Stokesian operator (Ref. 15) ;

$\Psi = \int_0^{\varpi} h_x \cdot 2\pi\varpi \, d\varpi$, total axial magnetic flux through circle of ϖ radius (Ref. 13) ;

$f = \int_0^{\varpi} j_x \cdot 2\pi\varpi \, d\varpi$, total axial volume current through same area (Ref. 13).

The solutions in spheroidal coordinates given in the previous section are seen to be examples of Ψ_{11} and Ψ_{21} . The equation

$$\mathcal{L}_{13} \Psi = (\mathcal{D} + c^2)\Psi = 0 \qquad (35)$$

has been considered in spheroidal coordinates also and separable solutions Ψ_{13} have been demonstrated (Ref. 16). Unfortunately, these solutions have not been tabulated but are related to the general spheroidal wave functions (Ref. 17). The equations for Ψ_{31} and Ψ_{33} do not seem to be separable.

The question of separability of a solution Ψ_{ij} in a general axi-symmetric coordinate system is probably an extension of separability theory for potential functions and their close relatives, solutions of the Helmholtz equation. Requirements for the separability of these equations have been given (Ref. 18) and the same requirements seem to hold for the generalized axi-symmetric potential (GASP) functions (Ref. 11). Little attention seems to have been directed to nonhomogeneous equations with the same operators. See the Appendix for further discussion.

IV. MAGNETOHYDRODYNAMIC SPHERICAL VORTEX

The previous sections dealt only with stationary fluid. A given magnetohydrostatic equilibrium configuration is disturbed if the fluid moves. If the fluid motion is very slow, we might imagine the disturbance was slight and that a new equilibrium was set up not very far from the hydrostatic one. For an arbitrary fluid motion, even for a perturbation analysis, one would have to calculate the flow disturbance to the magnetic field, which includes a boundary change. The next step is to solve the non-linear flow momentum equation (including the magnetic effect) which would probably introduce a further flow perturbation. With luck, the process would converge quickly but is likely to be difficult in detail.

By very great fortune, there is an exact flow solution of the full Navier-Stokes equation that introduces no disturbance to the magnetic vortex. This is the Hill-Hadamard spherical viscous vortex. The stream-function,

$$\psi = A \sin^2\theta \, (r^2 - r^4) , \tag{36}$$

has exactly the same form as the flux function Ψ , Eq. (12), for the spherical magnetic vortex. So the magnetic disturbance equation is (Ref. 2)

$$\mathbb{D}\Psi^* = \frac{R_m}{r^2 \sin\theta} \frac{\partial(\Psi, \psi)}{\partial(r, \theta)} = 0 , \tag{37}$$

using the star for the disturbance field. Thus the magnetic interaction term in the MHD vorticity equation is still exactly the same as for no motion. The full steady MHD vorticity equation, Eq. (7a) or (7b), is identically zero. Going back to the momentum equation, the scalar pressure will be increased by $\frac{\rho}{2} \vec{q} \cdot \vec{q}$ over the magnetohydrostatic value in the force balance.

Outside the spherical boundary we have no exact solution of the flow equation, but for small Re the terms are approximately in balance with the Hadamard-Rybzynski Stokes flow solution for the circulating sphere in uniform flow. Outside the sphere the conductivity is zero so there is no disturbance to the magnetic field there. The momentum balance is no different from the non-magnetic viscous case except for the magnetic pressure term. To the extent that the outer flow solution represents a possible steady flow there, the combination of ψ and Ψ gives a steady MHD vortex in dynamic equilibrium, Fig. 4. In fact, even if the outer flow were modified somewhat, by the presence of an outer boundary, say, but the change in outer flow pattern did not induce a change of streamlines within the vortex, the MHD configuration would still be in dynamic balance.

All the viscous boundary conditions are exactly satisfied at the spherical surface between the fluids. The normal velocity is identically zero, the tangential velocities in and out are continuous. The tangential and normal stress are also continuous across the boundary (Ref. 19). It is a curious fact that the pres or absence of an interface between the fluids does not affect the streamline pattern. An interface with given surface tension value merely contributes an additional term to the internal static pressure.

Steady spheroidal viscous vortices are also exact solutions of the full dynamic equations (Ref. 9). The linear Stokes flow equation in spheroidal coordinates can also be solved for the external flow (Ref. 20). Yet, unfortunately, the viscous boundary conditions cannot all be met simultaneously at the spheroidal surface (Ref. 21). Experimentally, steady approximately spheroidal viscous vortices are often observed when there is an interface between the fluids (Refs. 14, 22). The streamline flow within the vortex agrees qualitatively with the exact steady flow solution (Ref. 14). Even in the absence of an interface, if the translational velocity is sufficiently slow, some quasi-steady spheroidal vortices also display approximately

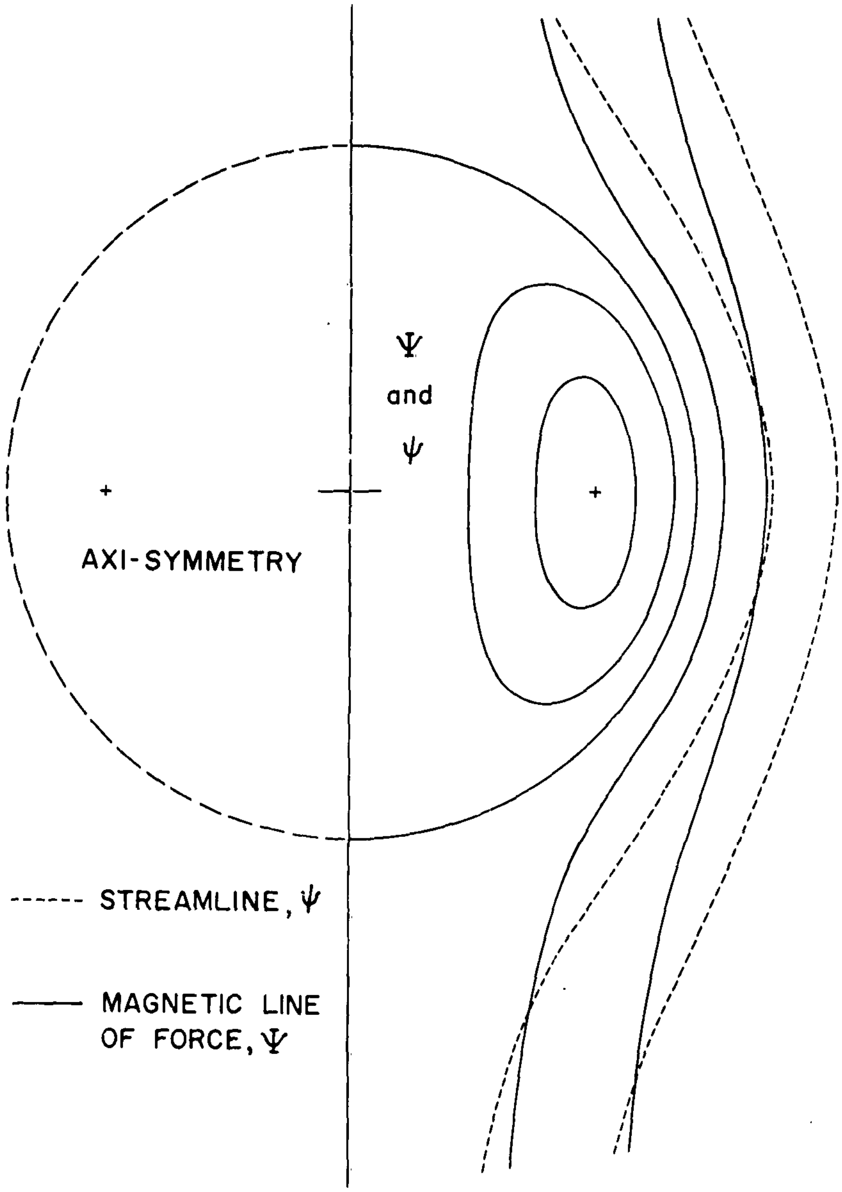

Fig. 4 SPHERICAL MHD VORTEX IN SPATIALLY UNIFORM MAGNETIC FIELD

the same flow pattern. As with the spherical vortex, it is possible that the spheroidal viscous flow patterns are duplicated in the magnetic vortex pattern. Because steady viscous vortices seem to exist only under the stabilizing influence of an interface, it might be inferred that steady MHD vortices could exist under similar conditions. However, questions of stability are far beyond this report, and, while it seems plausible, the existence of such vortices cannot be demonstrated here. The presence of a magnetic field often changes stability conditions markedly, sometimes tending to stabilize the conductive fluid motion and sometimes the reverse (Ref. 23).

It might be remarked that often viscosity is neglected, particularly for high $Re \to \infty$. Dropping the $1/Re$ term in the vorticity equation and dropping the viscous boundary conditions would allow an inviscid spheroidal MHD vortex in a uniform flow field with velocity vectors proportional to the magnetic vectors of the magnetohydrostatic vortices. There would be a vorticity jump at the boundary in this case, and however slight the viscosity, this is physically impossible.

The steady MHD spherical conducting vortex seems to refute Cowling's theorem, as it is usually stated, to the effect that a steady simply axi-symmetric motion of conductive fluid is impossible. Actually, Cowling's theorem applies to the case where the steady magnetic field reduces to zero at infinity (Ref. 24). In the present case, the boundary condition on the magnetic field is a spatially uniform field at infinity. It was shown that the magnetic vortex requires a certain time-dependent $H_0 = H_0(t)$. If it is possible to get a magnetohydrostatic vortex, it should be possible to get a MHD vortex with steady fluid motion.

It might be mentioned that the hydromagnetic equations, which imply inviscid fluid with infinite electrical conductivity, have been used by Agostinelli (Ref. 25) to derive a spherical vortex that moves along the applied magnetic field. However, the conditions of infinite conductivity and no viscosity are unrealistic when applied to real fluids. The use here of realistic viscosities and conductivity requires more stringent boundary conditions, but they can be met with the spherical viscous MHD vortex of fluid with arbitrary electrical conductivity.

V. PHYSICAL EXISTENCE OF MHD VORTICES

Useful theory in physical science depends upon the existence of a state of nature that corresponds to the theoretical hypotheses. Usually this state occurs rarely in our everyday experience but can often be set up under controlled conditions in the laboratory. There the reward of a theory self-consistent in hypotheses and approximations is the observation of behavior in accord with the theory. Until such a test is made, we can only conjecture on the possibility and its outcome. The conjectures here on a real spherical viscous MHD vortex are based on non-magnetic viscous flow theory on one hand and on observed magnetohydrodynamic behavior on the other.

The non-magnetic viscous flow theory for a rigid sphere in slow uniform motion has been tested in the laboratory. The observed steady motion in the Stokes flow range, $Re \leqslant 0.1$, conforms with the theory if the size of the fluid container is many times larger than the sphere. The effect of the container wall can be approximated theoretically too for the slow viscous flow range and the successful experimental check indicates the internal consistency of the viscous equations and the non-slip boundary conditions. See Fig. 5 of Ref. 26 for the conclusive evidence.

Similarly, if a moving fluid drop or bubble is a circulating sphere, the steady external flow and the viscous drag conform to theory within the same Re range (Ref. 26, Fig. 10). The fluid sphere in uniform motion may fail to have complete circulation due to some inhibitory effect of the interface, but quite often the spherical fluid vortex is observed. Even though the total drag coefficient is less for the fluid circulating sphere than for the rigid one at the same Reynolds number, up to one-third of the viscous dissipation occurs inside the sphere (it depends on the viscosity ratio of the fluids).

Consider some corresponding hydromagnetic cases. A solid non-ferrous metal (rigid, conducting) sphere moving steadily in a uniform aligned magnetic field within a non-conductive viscous fluid of the same permeability would have no MHD interaction.* The induced field inside the sphere will be uniform. No $\vec{q} \times \vec{h}$ currents would be induced either internally or externally by virtue of no motion across the field lines inside and no conductivity outside. The flow field would be identical to the non-magnetic flow field. It does not matter whether or not the magnetic field intensity is constant; there is no MHD interaction in the fluid. If the magnetic field intensity is time-dependent, there will be non-steady currents set up within the metal conducting sphere.

The flow field outside a fluid circulating sphere of conducting fluid (say, a mercury drop) moving uniformly along a uniform magnetic field will likewise have no <u>external</u> MHD interaction if the conductivity of the external fluid is zero. However, the fluid circulating sphere will have an <u>internal</u> MHD interaction. In general, there will be induced currents within the sphere whether the field intensity is constant or time-varying. This holds whatever the equilibrium configuration of streamlines and flux lines inside the sphere. The MHD configuration will probably be either a maximum or a minimum dissipation of energy. Consider the case of Section IV, where the streamlines and field lines within the sphere do coincide. Just as the Hill-Hadamard non-magnetic flow vortex solution represents the maximum viscous dissipation within the sphere with vorticity varying linearly with distance from the axis, the maximum ohmic dissipation to heat is given by the current distribution of the same form. Thus it seems likely that there is a steady MHD viscous vortex configuration with this maximum dissipation to heat. An experiment to test the theory is completely feasible.

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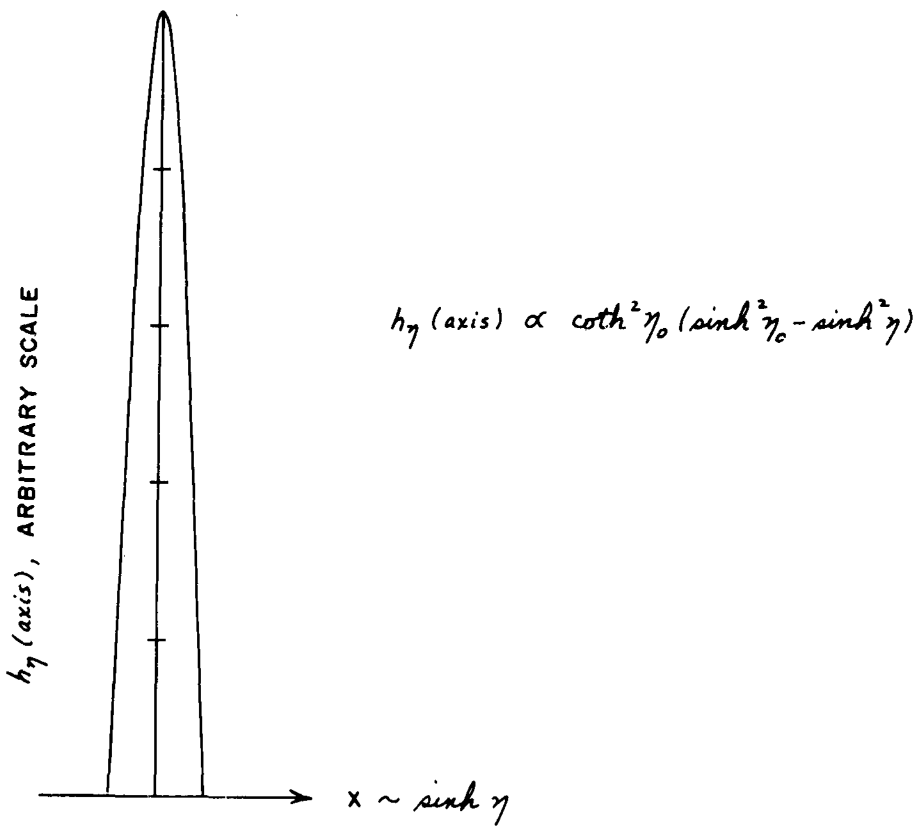
* There will be a static magnetic field disturbance if the permeabilities of the sphere and the fluid differ; see Ref. 3, Section III, and Fig. 2b.

The theory as derived in Section IV applies to an infinite fluid domain with a magnetic field of uniform intensity throughout. Obviously these conditions cannot be attained in a finite laboratory experiment. Such discrepancies of actuality and theory occur all the time and it is usually sufficient to consider a very large region where the desired conditions are satisfied. A large straight solenoid with accessory regulating equipment, cooling devices and so forth would provide the required magnetic field to a reasonable degree of accuracy. The limit to the flow field will change the outer flow lines slightly (the correction can be calculated) but will not change the non-magnetic flow lines within a circulating sphere. Thinking of the circulating sphere as a mercury drop, it is obvious that the flow lines within the opaque sphere in the MHD case will not be observable. However, a change in the flow pattern will be indirectly observable as a change in viscous drag. The magnetic field pattern associated with the moving drop can be inferred from small stationary magnetic pickup coils placed within the flow field, as they respond to the changing magnetic field. Even with poor sensitivity it should be possible to differentiate between the magnetic field patterns of Fig. 2a and Fig. 2b. Inevitably the magnetic probes will cause a disturbance to the MHD field but the extent of the disturbance should be revealed by changes of shape and velocity of the drop with the probes in place compared to the drop behavior in their absence.

The MHD spherical vortex analysis applies only to a low Reynolds number flow. However, if, as in the spherical vortex example, the flow lines within a magnetic surface tend to align themselves with the magnetic flux lines, the magnetohydrostatic flux configurations may also be nearly the MHD flux configurations provided the flow situation is relatively steady. In some plasma experiments a toroidal arrangement of magnetic fields is used to confine the hot plasma in the central portion of a toroidal tube. These fields usually possess both axial and azimuthal components. Although the MHD configurations are not steady, the observed fields seem to progress through a series of axi-symmetric configurations approximately in accord with a set of magnetohydrostatic equilibria calculated by means of Ψ_{23} (see Ref. 5).

Recently, an axi-symmetric configuration yielded some fairly long-lived "plasma vortex rings" (Ref 27). The plasmoids were actually roughly spheroidal in shape and had internal vortex motion without azimuthal components. They were followed by a second structure that appeared to have azimuthal motion. The measurements of magnetic field on the center line of the plasmoid are not inconsistent with the magnetic spheroidal vortex, Fig. 5. However, the rings were moving with considerable relative velocity, 5×10^6 cm/sec, and had internal circulation, so it is probably correct to infer, as the author of the article did, that the streamlines and field lines were identical within the vortex. This is not quite the same as the statement, "possesses the same magnetic field and velocity distributions", which seems inconsistent with translational motion in the same direction for all the rings while the magnetic field reversed from one vortex ring to the next, Fig. 5b. Note that the driving magnetic field was time-dependent, Fig. 5b. Perhaps it will soon be possible to describe the structure within the plasmoids in more detail to see if there is a magnetic torus of small cross-section (true vortex ring) or a spheroidal vortex.

Finally, there sometimes appears, as a result of lightning discharge, a glowing ball often described as undergoing rapid internal rotation as well as traveling parallel to the earth's surface, though some distance above it (Refs. 28 and 29). This "ball lightning" appears hot and brilliant, lasts on the average 3 or 4 seconds, but may last many minutes, often disappearing with a loud explosion (implosion). Some people have attributed this phenomenon to a free-floating plasma mass in an external r.f. electric field. Such plasmas have been produced in the laboratory, but the high frequency field does not penetrate the plasma



(a)

Fig. 5a THEORETICAL VALUE OF AXIAL VECTOR COMPONENT IN SPHEROIDAL MAGNETIC VORTEX

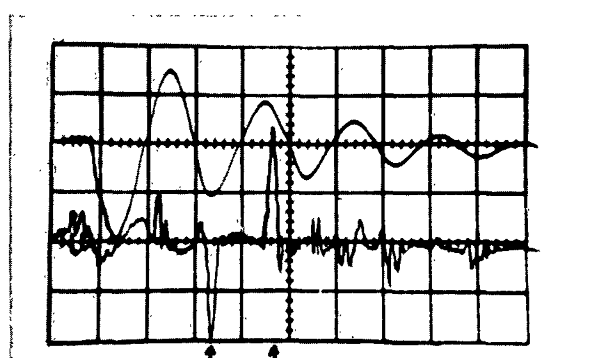


FIG. 3. Dual-beam oscilloscope trace. Top trace shows the punch-coil magnetic field vs time. Bottom trace is the signal on a magnetic pickup coil placed in the center of the drift tube at the center of the window used for the exposures shown in Figs. 1 and 2.

Fig. 5b MEASURED VALUE OF AXIAL MAGNETIC VECTOR COMPONENT IN "PLASMA VORTEX-RINGS" (Reference 27) and it differs from the ball lightning in various respects. There is a strong possibility that the natural phenomenon is a true magnetic vortex with internal currents. The observation of "donut" forms as well as balls suggests it is closely allied to the laboratory produced plasmoids described above.

Thus there is evidence that real MHD vortices can exist in nature under certain conditions. It seems highly probable that a steady viscous MHD vortex can exist in uniform translational motion through an aligned magnetic field at low Re. The theoretical model of the steady spherical MHD vortex which satisfies all reasonable boundary conditions for real fluids may be expected to describe the MHD field.

VI. SUMMARY

Vorticity is an important property of ordinary viscous fluids, especially in the vicinity of moving objects. It seemed natural to assume that vorticity would also be important for the magnetohydrodynamic (MHD) flow of incompressible conductive fluid in the presence of magnetic fields. As in previous work on MHD fields (Refs. 1, 2, 3, 11), the assumption of a simple axi-symmetric configuration allows some mathematical simplification without sacrificing any physical reality: The complete MHD field of simple axi-symmetry can be reduced to the solution of two scalar equations in terms of the Stokes streamfunction and an analogous magnetic flux function (Ref. 2). These partial differential equations are non-linear and intimately coupled so no general solution can be obtained to fit all MHD problems. However, they are in a form that allows complete solution for certain problems. In this report these equations were utilized to study some MHD vortices, dealing with a finite body of conductive fluid in a magnetic field.

There are conditions under which only the magnetic field has rotationality, corresponding to free volume-current flow, while the flow field is stationary. Some magnetohydrostatic configurations have been studied before through a formal mathematical analogy with the steady rotational flow of a perfect fluid. The equations of magnetohydrostatic equilibria were re-examined here and some solutions given extending the previous results. Specifically, an infinite class of magnetic vortices in a uniform magnetic field was demonstrated where the boundary shape is spheroidal, prolate or oblate of any fineness ratio.

A truly magnetohydrodynamic vortex solution for a moving flow field is usually intractable because of the non-linear interaction between the magnetic field and the flow field. It was shown that there is an exceptional case, the spherical Hill-Hadamard viscous conducting vortex in a uniform but time-varying magnetic field. This MHD flow solution permits all realistic boundary conditions for the viscous fluid and the magnetic field components to be satisfied simultaneously. The possibility of an experimental verification of the theory was discussed, along with the relation of the theory to observed phenomena such as "ball lightning" and "plasma vortex rings".

APPENDIX

Consideration of Magnetohydrostatic Equations

The set of equations, Table I,

$$\{ \mathcal{L}_{ij} \Psi = F_{ij}(\varpi) \} , \qquad \text{(A-1)}$$

all represent possible magnetohydrostatic equilibria. As in other problems with particular geometric boundaries, it is often advantageous to have coordinate systems with the same geometry. Therefore, this section will discuss solutions $\{ \Psi_{ij} \}$ in various axi-symmetric coordinate systems.

$\boxed{\Psi_{11}}$ In cylindrical polar coordinates (x, ϖ, φ)

$$\mathcal{L}_{11} \Psi = \mathbb{D} \Psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} - \frac{1}{\varpi} \frac{\partial}{\partial \varpi} \right) \Psi = 0. \qquad \text{(A-2)}$$

This equation is a generalized axially symmetric potential equation. It is closely related to the ordinary axi-symmetric potential equation,

$$\nabla^2_{axi} \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} \right) \phi = 0, \qquad \text{(A-3)}$$

which has been studied in various axi-symmetric coordinate systems (Ref. 18). The conditions for separability of Eq. (A-3) have been given (Ref. 18), and the conditions for the generalized equations seem to be exactly the same (Ref. 11). In fact, for Eq. (A-2) the solution can be formulated in terms of the potential ϕ . The separated general solutions for Eq. (A-2) have been given in spherical polar, oblate spheroidal, prolate spheroidal, toroidal, dipolar and peri-polar coordinates (Refs. 3, 11). A number of more esoteric coordinates, such as cap cyclides (Ref. 30), have also been used to discuss the potential equation and, no doubt, the solution for the Stokesian operator would follow from a similar treatment.

$\boxed{\Psi_{12}}$ The equation for Ψ_{12},

$$\mathcal{L}_{12}\Psi = -\frac{b^2}{2} = \mathcal{D}\Psi, \tag{A-4}$$

is a nonhomogeneous equation with the same Stokesian operator. Therefore, a particular solution is required to add to the complementary solution above. In cylindrical polar coordinates, it is easily shown that

$$\Psi_{12}(\varpi) = -\frac{b^2}{4}\varpi^2 \ln \varpi \tag{A-5}$$

is such a solution. In spherical polar coordinates, this becomes

$$\Psi_{12}(r,\theta) = -\frac{b^2}{4} r^2 \sin^2\theta \ln (r \sin\theta), \tag{A-6}$$

which can be checked as a solution fairly easily using $\mathcal{D}$ written in these coordinates, Section I. Similarly, the solution in any axi-symmetric coordinate system (α, β, φ) can be written as

$$\Psi_{12}(\alpha,\beta) = -\frac{b^2}{4}\varpi^2(\alpha,\beta) \ln [\varpi(\alpha,\beta)]. \tag{A-7}$$

$\boxed{\Psi_{13}}$ The equation for Ψ_{13},

$$\mathcal{L}_{13}\Psi = (\mathcal{D} + c^2)\Psi = 0, \tag{A-8}$$

contains the operator ($\mathcal{D} + c^2$) which is rather similar to the Helmholtz operator ($\nabla^2_{axi} + c^2$) which has previously been considered in a number of axi-symmetric coordinate systems (Ref. 18). Separability of the Helmholtz equation,

$$(\nabla^2_{axi} + c^2)\phi = 0, \tag{A-9}$$

requires one more condition, in general, than separability of the Laplacian. However, for all those rotational coordinate systems where the map in a meridian plane is a conformal mapping from the x, y map in that plane, the extra condition is always satisfied (Ref. 18). The common axi-symmetric coordinate systems do allow separation of the Helmholtz equation (A-9) and separation of Eq. (A-8) as well. Separated solutions Ψ_{13} in cylindrical and spherical polar coordinates, oblate and prolate spheroidal coordinates have been given in the literature. These two-dimensional solutions all seem to correspond to types of three-dimensional wave functions.

$\boxed{\Psi_{21}}$ The nonhomogeneous equation for Ψ_{21},

$$\mathcal{L}_{21}\Psi = B\varpi^2 = \mathcal{D}\Psi, \tag{A-10}$$

requires only a particular solution to add to the complementary solution Ψ_{11}. In the cylindrical polar coordinate solution it is easy to see that such a solution is

$$\Psi_{21}(\varpi) = \frac{B}{8}\varpi^4. \tag{A-11}$$

This can be transformed to other coordinate systems and gives the somewhat surprising result that a simple separated particular solution exists in the spheroidal coordinate systems even though the differential equation itself is not simply separable. In the spherical polar system another particular solution is found to be

$$\Psi_{21}(r,\theta) = r^4 \sin^2\theta \, \frac{B}{8}. \tag{A-12}$$

Of course, either of these particular solutions, transformed to the appropriate coordinates, can be used with the general solution Ψ_{11} to satisfy boundary conditions.

$\boxed{\Psi_{22}}$ The nonhomogeneous equation for Ψ_{22} is

$$\mathcal{L}_{22}\Psi = \mathcal{D}\Psi = -\frac{b^2}{2} + B\varpi^2. \tag{A-13}$$

The particular solution is just the sum of a particular solution above and Ψ_{12}. That is,

$$\Psi_{22}(\varpi) = \frac{B}{8}\varpi^4 - \frac{b^2}{4}\varpi^2 \ln\varpi, \tag{A-14}$$

while in other coordinate systems $\varpi(\alpha,\beta)$ is substituted.

$\boxed{\Psi_{23}}$ The equation for Ψ_{23} is also nonhomogeneous,

$$\mathcal{L}_{23}\,\Psi = (\mathcal{D} + c^2)\,\Psi = B\varpi^2, \tag{A-15}$$

but with the "generalized Helmholtz" operator. This presents a slightly different particular solution but also an easy one.

$$\Psi_{23}(\varpi) = \frac{B}{C^2}\,\varpi^2, \tag{A-16}$$

with similar simple transformed solutions in other coordinate systems.

$\boxed{\Psi_{31}}$ The operator of the equation for Ψ_{31},

$$\mathcal{L}_{31}\,\Psi = (\mathcal{D} + C\varpi^2)\,\Psi = 0, \tag{A-17}$$

that is, $\mathcal{D} + C\varpi^2$ seems to require a new type of solution that doesn't fit in with GASP (generalized axially symmetric potentials) and wave functions. The equation can be separated easily in the cylindrical polar coordinate system.

Let

$$\Psi = F(\varpi)\,G(x). \tag{A-18}$$

Equation (A-17) becomes

$$\frac{G''(x)}{G(x)} + \frac{1}{F(\varpi)}\left[F''(\varpi) - \frac{1}{\varpi}F'(\varpi) - \alpha^2\varpi^2 F(\varpi)\right] = 0$$

where $\alpha^2 = -C$. Using separation constant $K^2 = \alpha^2 k^2$

$$\frac{G''}{G} = -K^2 \qquad\qquad \alpha^2 k^2 F = F'' - \frac{1}{\varpi}F' - \alpha^2\varpi^2 F$$

$$G = e^{\pm iKx} \qquad\qquad 0 = F'' - \frac{1}{\varpi}F' - \alpha^2(\varpi^2 + k^2).$$

Changing the independent variable to z,

$$z = \varpi^2 + k^2,$$

the equation for F becomes,

$$(z - k^2)\frac{d^2F}{dz^2} - \frac{\alpha^2}{4}zF = 0,$$

which is a convenient form to develop the recurrence relations between coefficients of an infinite series representation of F. Let $F = \sum_{n=0}^{\infty} a_n z^n$;

then $$\sum_{n=0}^{\infty} a_n \left[(z - k^2) n(n-1) z^{n-2} - \frac{\alpha^2}{4} z^{n+1} \right] = 0,$$

or $$\sum_{m=0}^{\infty} \left[(m+2)(m+1) a_{m+2} - k^2 (m+3)(m+2) a_{m+3} - \frac{\alpha^2}{4} a_m \right] z^{m+1} = 0.$$

Thus
$$\Psi_{31}(x, \varpi) = \sum_{m=0}^{\infty} a_m (c, k^2) (\varpi^2 + k^2)^{m+1} \{ b(k) e^{i\alpha k x} + c(k) e^{-i\alpha k x} \}. \qquad \text{(A-19)}$$

Of course, a similar solution in descending powers of ϖ can be developed for regions far from the origin.

When $k = 0$, the solution is independent of x and a closed expression $F_0(\varpi)$ is obtained:

$$\Psi_{31} = F_0(\varpi) = \exp\left(\pm \frac{\alpha}{2} \varpi^2 \right). \qquad \text{(A-20)}$$

The equation (A-17) does not appear to be separable in any other rotational coordinate system, even the spherical polar one:

$$\mathcal{L}_{31} \Psi = \left\{ \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2} \frac{\partial^2}{\partial \mu^2} - \alpha^2 r^2 (1-\mu^2) \right\} \Psi = 0 \qquad \text{(A-21)}$$

where $\mu = \cos\theta$.

$\boxed{\Psi_{32}}$ The nonhomogeneous equation for Ψ_{32},

$$\mathcal{L}_{32} \Psi = -\frac{b^2}{2} = \mathcal{L}_{31} \Psi = (\mathcal{D} - \alpha^2 \varpi^2) \Psi, \quad (\alpha^2 = -C) \qquad \text{(A-22)}$$

requires a particular solution. Assuming the particular solution to be a function of ϖ alone, and an ascending series

$$\Psi_{32} = P(\varpi) = \sum_n \left[a_n \varpi^n \ln \varpi + b_n \varpi^n \right]. \tag{A-23}$$

Eq. (A-22) can be reduced to

$$\sum_{\substack{m=0 \\ (\text{even})}}^{\infty} 2a_2 + \left\{ (m+4)(m+2)\, a_{m+4} - \alpha^2 a_m)(\ln \varpi)\varpi^{m+2} \right\} + \varpi^{m+2}\left[2a_{m+4} - (m+4)(m+2)\, b_{m+4} \right] = -\frac{b^2}{2}.$$

leading to the relations between coefficients:

$$a_2 = -\frac{b^2}{4}.$$

$$a_{m+4} = \frac{\alpha^2}{(m+4)(m+2)}\, a_m.$$

$$b_{m+4} = \frac{2}{(m+4)(m+2)}\, a_{m+4}.$$

When $\alpha = 0$ this reduces to the particular solution Ψ_{12} given above. The solution can be transformed to other coordinate systems.

$\boxed{\Psi_{33}}$ The final equation,

$$\mathcal{L}_{33}\, \Psi = (\mathcal{D} - \alpha^2 \varpi^2 + c^2)\, \Psi = 0, \tag{A-24}$$

can be separated in the cylindrical polar coordinate system, if no other. With

$$\Psi_{31}(x, \varpi) = \underline{F}(\varpi)\, \underline{G}(x) \tag{A-25}$$

Eq. (A-24) becomes

$$\frac{\underline{G}''(x)}{\underline{G}'(x)} + \frac{1}{\underline{F}(\varpi)}\left[\underline{F}''(\varpi) - \frac{1}{\varpi}\underline{F}'(\varpi) - \alpha^2 \varpi^2 \underline{F} + c^2 \underline{F} \right] = 0.$$

With separation constant $K^2 = \alpha^2 (k^2 - \frac{c^2}{\alpha^2}) = \alpha^2 k_1^2$,

$$\frac{\underline{G}''(x)}{\underline{G}(x)} = -K^2; \qquad \alpha^2 (k^2 - \tfrac{c^2}{\alpha^2})\underline{F} = \underline{F}'' - \tfrac{1}{\varpi}\underline{F}' - \alpha^2 (\varpi^2 - \tfrac{c^2}{\alpha^2})\underline{F},$$

$$0 = \underline{F}'' - \tfrac{1}{\varpi}\underline{F}' - \alpha^2 (\varpi^2 + k^2)\underline{F}.$$

$$\underline{G} = e^{\pm i\alpha k_1 x} ; \qquad \underline{F} = F(\varpi).^{\text{(above)}}$$

These separated equations reduce to essentially the ones above for Ψ_{31}.

Note that the solutions on the central cross rather than the corners of the matrix $\{\Psi_{ij}\}$ will probably be more significant in fitting boundary conditions on a magnetic surface that encloses conductive fluid. Note also either Ψ_{21}, Ψ_{22} or Ψ_{23} may give a solution $\Psi = 0$ on a given surface, but they correspond to different axial current distributions.

For possible steady magnetohydrodynamic solutions, Ψ_{21} is probably most significant physically. If the streamfunction ψ and flux function Ψ are aligned, there is no disturbance to the magnetic vortex field and the vorticity distributions corresponding to these solutions satisfy not only the non-linear inertial requirement for steady flow but also the corresponding viscous terms. All individual terms below are identically zero when $f \equiv 0$.

Flow Vorticity Equations:

$$\frac{\partial}{\partial t} D\psi + \sin\theta \frac{\partial\left(\psi, \frac{D\psi + ff'}{r^2 \sin^2\theta}\right)}{\partial(r,\theta)} = \frac{1}{Re} D^2\psi - \frac{M^2(t)}{Re\,R_m} \sin\theta \frac{\partial\left(\Psi, \frac{D\Psi + ff'}{r^2\sin^2\theta}\right)}{\partial(r,\theta)} .$$

$$\frac{\partial f}{\partial t} + \frac{1}{r^2 \sin\theta} \frac{\partial(\psi, f)}{\partial(r,\theta)} = -\frac{1}{Re} Df - \frac{M^2(t)}{Re\,R_m} \frac{\partial(\Psi, f)}{\partial(r,\theta)} .$$

For $f \neq 0$, the azimuthal components are not in steady dynamic equilibrium.

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The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

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